A highly accurate DFT-based parameter estimator for complex exponentials

Sam Reisenfeld
Macquarie University

Paper

A highly accurate DFT-based parameter estimator for complex exponentials

Jeffrey Tsui and Sam Reisenfeld

Abstract— A highly accurate DFT-based complex exponential parameter estimation algorithm is presented in this paper. It will be shown that for large number of samples and high signal to noise ratio (SNR), the phase estimation error variance performance is only 0.0475 dB above the Cramer-Rao lower bound (CRLB) for phase estimation with unknown frequency and phase. The amplitude estimation error variance performance was found to lay on the CRLB for amplitude estimation. Exact phase and amplitude estimation can be achieved in the noiseless case with this algorithm. The algorithm has low implementation computational complexity and is suitable for numerous real time digital signal processing applications.

Keywords— frequency estimation, phase estimation, amplitude estimation, DFT-based parameter estimation, spectral estimation, digital signal processing algorithm, complex exponential parameter estimation.

1. Introduction

Frequency, phase and amplitude estimation of a complex exponential is classical problem in statistical signal processing. The precision of the phase and amplitude estimate is directly related to the accuracy of the frequency estimate. There are two major classes of complex exponential frequency estimation algorithm in the existing literature. The first class is the classical discrete Fourier transform (DFT) and phase averager based frequency estimation algorithms such as [3] and [4]. This class of estimation algorithm is very computationally efficient, however they suffer from poor error performance at low signal to noise ratio (SNR). The second class is the parametric frequency estimation algorithms such as the very popular MUSIC and ESPRIT algorithms [5, 6]. This class of estimation algorithm has very good error performance across a wider range of SNR, however, they are very computationally intensive and not suitable for real-time application. Obviously, it would be ideal if there exists a complex exponential parameter estimator that is both computationally efficient and has good performance at low SNR.

In 2003, Reisenfeld and Aboutanios [2] discovered a frequency estimator that can precisely estimating the frequency of a complex exponential in an iterative manner by applying contraction mapping method on two modified DFT coefficients. In a subsequence publication in 2004, Reisenfeld [1] further enhanced the previous published algorithm by improving the convergence property of the estimator. It can be shown that this particular frequency estimator can yield a frequency estimate that is 0.0633 dB of the Cramer-Rao lover band (CRLB) with approximately $N\log_2 N + 4N$ complex multiplications, where N is the number of samples used representing the signal.

In this paper, it will be shown that combining the said frequency estimator with maximum likelihood (ML) phase and amplitude estimators yields a highly accurate parameter estimator for complex exponentials. In the noiseless case, it is possible to obtain the exact phase and amplitude estimates with this estimator. In additive white Gaussian noise (AWGN) channel, the said estimates approach very close the their respective CRLB at relative high SNR. The relationship between the number of samples, N, and the operating point of the parameter estimator in terms of SNR will be given. It will also be shown that it is possible to use the said relationship to further optimise the computation complexity of the estimator.

The rest of the paper will be organised as follows: Section 2 introduces the proposed DFT-based parameter estimator, Section 3 will provide the performance analysis of the proposed parameter estimator. Section 4 will describe further enhancements that can be made to the original frequency estimator in [1] to reduce its computation complexity. Section 5 contains the simulated results of the performance of the proposed algorithm and this will be followed by the conclusion.

2. The DFT-based parameter estimator

The proposed DFT-based parameter estimator involves a two stage process. First, the frequency of the received carrier is estimated by the frequency estimator as described in [1]. The frequency estimate obtained is then used to eliminate the frequency component of the carrier, leaving only the phase and amplitude component to be estimated by the ML phase and amplitude estimator.

2.1. The DFT-based complex exponential frequency estimator

Consider a complex exponential, $r[n]$, with amplitude, A_c, frequency, $f_c \in [0, f_s)$, and phase $\theta_c \in [0, 2\pi)$. Mathematically, $r[n]$ can be represented as

$$r[n] = A_c e^{j(2\pi f_c n T_s + \theta_c)} + \eta[n], \tag{1}$$

where $n = 0, 1, 2, ..., N-1$, $T_s = 1/f_s$ is the sampling period, and $\eta[n]$ is a sequence of independent complex Gaussian variables with mean of zero and variance σ^2.

In noiseless condition, the magnitude spectra the DFT of Eq. (1) will have even symmetry about its frequency. The recursive algorithm as described in [1] exploits this condition by employing a discriminate that works on a contraction principle in minimizing the difference in the magnitude of two modified DFT coefficients which are plus and minus half a DFT bin away from an estimated frequency. Due to the even symmetric nature of the magnitude spectra, the difference in the magnitude of the two modified DFT coefficients will eventually reduced to zero in the noiseless case as the estimated frequency approaches the true frequency with the increasing number of recursions of the algorithm. The frequency can then be estimated as the mean of the frequencies of the modified DFT coefficients at which difference in their magnitude equals to zero.

Summarizing the DFT-based frequency estimator in [1]:

1. Perform a coarse frequency estimate such as the one described in Rife and Boorstyn [3], in which $\{r[n]\}_0^{N-1}$ is the input to a N point complex DFT and a peak search is done on the magnitudes of the DFT output coefficients, to obtain the initial frequency estimate, $\hat{f}_0$. This estimate is obtained by, $\hat{f}_0 = k_{\max} f_s / N$, where $k_{\max}$ is the index of the maximum magnitude DFT output coefficient.

2. Calculate the modified DFT coefficients α_m and β_m defined by

$$\alpha_m = \sum_{n=0}^{N-1} r[n]\, \mathrm{e}^{-\mathrm{j}2\pi n\left(\hat{f}_m T_s - \frac{1}{2N}\right)}, \tag{2}$$

$$\beta_m = \sum_{n=0}^{N-1} r[n]\, \mathrm{e}^{-\mathrm{j}2\pi n\left(\hat{f}_m T_s + \frac{1}{2N}\right)}. \tag{3}$$

3. Calculate the discriminate D_m defined as

$$D_m = \frac{|\beta_m| - |\alpha_m|}{|\beta_m| + |\alpha_m|}. \tag{4}$$

4. Calculate the new adjusted frequency with the formula:

$$\hat{f}_{m+1} = \hat{f}_m + \frac{1}{\pi}\tan^{-1}\left[D_m \tan\left(\frac{\pi}{2N}\right)\right] f_s. \tag{5}$$

5. Perform Steps 2–4 recursively for $m = 1, 2, 3, \ldots, M-1$.

2.2. Combining the frequency estimator with ML phase and amplitude estimator

Using the frequency estimate $\hat{f}_m$ obtained from the frequency estimator described above, it is possible to eliminate the frequency component of $r[n]$ by multiplying it with the conjugate of the complex exponential with a frequency of $\hat{f}_m$. Denoting $r_{A,\theta}[n]$ as the result of the multiplication we have

$$\begin{aligned} r_{A,\theta}[n] &= \left(A\mathrm{e}^{\mathrm{j}(2\pi f_c n T_s + \theta_c)} + \eta[n]\right)\cdot \mathrm{e}^{-\mathrm{j}2\pi \hat{f}_m n T_s} \\ &= A\mathrm{e}^{\mathrm{j}(2\pi \varepsilon n T_s + \theta_c)} + \eta[n], \end{aligned} \tag{6}$$

where $\varepsilon = f_c - \hat{f}_m$.

It was shown in [7] that the ML phase estimate of a complex exponential is obtained by taking its arctangent. Taking the arctangent of $\sum_{n=0}^{N-1} r_{A,\theta}[n]$ yields

$$\begin{aligned} \hat{\theta}_{c,\hat{f}_m} &= \tan^{-1}\left(\frac{\mathrm{Im}\left[\sum_{n=0}^{N-1}\left(A\mathrm{e}^{\mathrm{j}(2\pi\varepsilon n T_s+\theta_c)} + \eta[n]\right)\right]}{\mathrm{Re}\left[\sum_{n=0}^{N-1}\left(A\mathrm{e}^{\mathrm{j}(2\pi\varepsilon n T_s+\theta_c)} + \eta[n]\right)\right]}\right) \\ &= \tan^{-1}\left(\frac{\sum_{n=0}^{N-1} A\sin(2\pi\varepsilon n T_s + \theta_c) + \eta_s}{\sum_{n=0}^{N-1} A\cos(2\pi\varepsilon n T_s + \theta_c) + \eta_c}\right), \end{aligned} \tag{7}$$

where

$$\eta_c = \mathrm{Re}\left(\sum_{n=0}^{N-1}\eta[n]\right),\quad E[\eta_c] = 0,\quad Var[\eta_c] = \frac{N\sigma^2}{2}, \tag{8}$$

$$\eta_s = \mathrm{Im}\left(\sum_{n=0}^{N-1}\eta[n]\right),\quad E[\eta_s] = 0,\quad Var[\eta_s] = \frac{N\sigma^2}{2}, \tag{9}$$

and $\hat{\theta}_{c,\hat{f}_m}$ is the phase estimate based upon the estimated frequency.

It was also shown in [7] that the ML amplitude estimate of a complex exponential is obtained by taking its absolute value. Taking the absolute value of $\sum_{n=0}^{N-1} r_{A,\theta}[n]$ yields

$$\begin{aligned} \hat{A}_{\hat{f}_m} &= \frac{1}{N}\sqrt{\mathrm{Re}\left[\sum_{n=0}^{N-1}\left(A\mathrm{e}^{\mathrm{j}(2\pi\varepsilon n T_s+\theta_c)}+\eta[n]\right)\right]^2 + \mathrm{Im}\left[\sum_{n=0}^{N-1}\left(A\mathrm{e}^{\mathrm{j}(2\pi\varepsilon n T_s+\theta_c)}+\eta[n]\right)\right]^2} \\ &= \frac{1}{N}\sqrt{\left[\sum_{n=0}^{N-1} A\cos(2\pi\varepsilon n T_s) + \eta_c\right]^2 + \left[\sum_{n=0}^{N-1} A\sin(2\pi\varepsilon n T_s) + \eta_s\right]^2}. \end{aligned} \tag{10}$$

Notice the amplitude estimation is not affected by the phase angle of the complex exponential θ_c.

From Eqs. (7) and (10), it is obvious that a highly accurate frequency estimator can assist in reducing the error and improve on the accuracy of the phase estimation. Hence the described frequency estimator is very suitable for this joint estimation of phase and amplitude due to its superior error variance performance.

3. Performance of the parameter estimator

3.1. Frequency estimator

The author in [1] has proven this algorithm only requires two iterations for the variance of the frequency estimate $\hat{f}_2$ to converge to less than or equal to 0.063 dB above the CRLB for frequency estimation.

3.2. Phase estimator

Assuming high SNR, using Taylor series expansion as described in [8] the variance of Eq. (7) was found to be

$$Var\left[\hat{\theta}_{c,\hat{f}_m}\right] = \frac{N^2(N-1)^2\sin^2\left(\frac{\pi}{2N}\right)\tan^2\left(\frac{\pi}{2N}\right)+2}{4\rho N}, \quad (11)$$

where ρ, the SNR equals to

$$\rho = \frac{A^2}{\sigma^2}. \quad (12)$$

Since this algorithm jointly estimates the frequency and phase of the observed signal, it is appropriate to compare the performance of the phase estimator to the CRLB of phase estimation with unknown frequency and phase which is given by [7]

$$CRLB_{joint}(\theta) = \frac{(2N-1)}{N\rho(N+1)}. \quad (13)$$

Therefore

$$\frac{Var\left[\hat{\theta}_{c,\hat{f}_m}\right]}{CRLB_{joint}(\theta)} = \frac{\left(N^2(N-1)^2\sin^2\left(\frac{\pi}{2N}\right)\tan^2\left(\frac{\pi}{2N}\right)+2\right)(N+1)}{4(2N-1)}. \quad (14)$$

For large ρ and large N, it can be shown,

$$\lim_{N\to\infty} 10\log_{10}\left(\frac{Var\left[\hat{\theta}_{c,\hat{f}_m}\right]}{CRLB_{joint}(\theta)}\right) = 10\log_{10}\left(\frac{\pi^4}{128}+\frac{1}{4}\right) = 0.0475 \text{ dB}. \quad (15)$$

Figure 1 shows the convergence property of $Var\left[\theta_{est,f}\right]$ to the CRLB as a function of the number of samples N. It can be seen that variance of the phase estimator deviates from the CRLB as the number of samples N increases and approaches the asymptotic limit of 0.0475 dB. This due to the convergence behavior of the frequency estimator as stated in [1] where for small N, less information is discarded for not using all the DFT coefficients hence the performance degradation is less than when N is large.

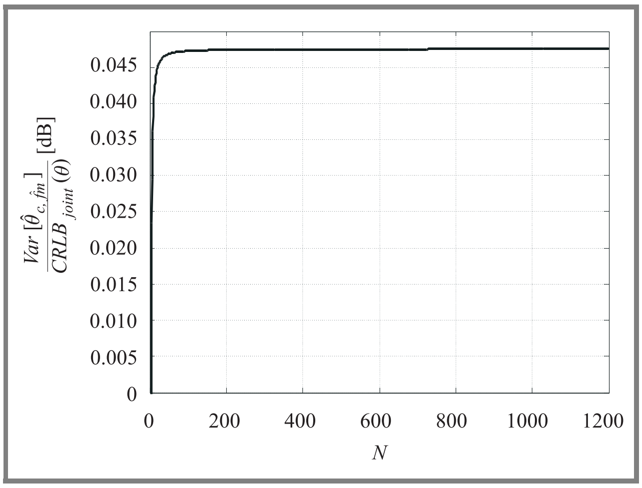

Fig. 1. $\frac{Var\left[\hat{\theta}_{c,\hat{f}_m}\right]}{CRLB_{joint}(\theta)}$ in dB as a function of number of samples, N.

3.3. Amplitude estimator

Assuming high SNR, using Taylor series expansion as described in [8] the variance of Eq. (10) was found to be

$$Var\left[\hat{A}_{\hat{f}_m}\right] = \frac{\sigma^2}{2N}, \quad (16)$$

which agrees with the CRLB derived by Rife and Boorstyn in [3].

4. Further enhancements

As describe in Subsection 2.1, the frequency estimator in [1] requires an initial frequency estimate obtained by performing a fast Fourier transform (FFT) operation on the signal samples. Since FFT requires $N\log_2 N$ complex multiplications, it is desirable to keep the number of samples N as low as possible for the initial coarse estimate. However, the CRLB for frequency estimation monotonically decreases with the increasing number of samples used for the estimation at a fixed SNR. Hence, reducing the number of samples used for the frequency estimate may not produce an estimate that will meet the required accuracy.

One way to reduce the complexity of the estimator and obtain the required accuracy is to modify the frequency estimator so that it beings the initial coarse frequency estimation with a low number of samples and dynamically increase the number of samples for each iteration of frequency estimation algorithm. The modified version of the DFT-based frequency estimator is summarised as follows:

Denoting N_i, where $i = 0,1,2,3,...,I$, as the number of samples used in the ith pass of the frequency estimation algorithm. Note that the number of samples, N_i, for each pass must satisfy the following relationship $N_0 \le N_1 \le N_2 \le ... \le N_I$ and $N_I = N$ which is the number of samples required to obtain the desire accuracy.

1. Perform a coarse frequency estimate such as the one described in Rife and Boorstyn [3], in which $\{r[n]\}_0^{N_0-1}$ is the input to a N_0 point complex DFT

and a peak search is done on the magnitudes of the DFT output coefficients, to obtain the initial frequency estimate, $\hat{f}_0$. This estimate is obtained by, $\hat{f}_0 = k_{\max} f_s / N_0$, where $k_{\max}$ is the index of the maximum magnitude DFT output coefficient.

2. Perform the ith pass of the frequency estimator which consists of the following steps:

   2.1. Recursion is started at $m = 0$.

   2.2. Set $N_i = N_0$.

   2.3. Calculate the modified DFT coefficients α_m and β_m defined by

$$\alpha_m = \sum_{n=0}^{N_i-1} r[n]\, \mathrm{e}^{-\mathrm{j}2\pi n\left(\hat{f}_m T_s - \frac{1}{2N_i}\right)}, \tag{17}$$

$$\beta_m = \sum_{n=0}^{N_i-1} r[n]\, \mathrm{e}^{-\mathrm{j}2\pi n\left(\hat{f}_m T_s + \frac{1}{2N_i}\right)}. \tag{18}$$

   2.4. Calculate the discriminate D_m defined as

$$D_m = \frac{|\beta_m| - |\alpha_m|}{|\beta_m| + |\alpha_m|}. \tag{19}$$

   2.5. Calculate the new adjusted frequency with the formula:

$$\hat{f}_{m+1} = \hat{f}_m + \frac{1}{\pi}\tan^{-1}\left[D_m \tan\left(\frac{\pi}{2N_i}\right)\right] f_s. \tag{20}$$

   2.6. Perform Steps 2.3–2.5 recursively for $m = 1, 2, 3, \ldots, M-1$.

3. Set the value of $\hat{f}_0$ to the value of $\hat{f}_m$ as found in Step 2.5. Repeat Step 2 for $N_i = N_1$, $N_i = N_2$, ..., $N_i = N_I$.

Choosing the value of N_i's. Thus far, the discussion has been on the reducing the number of samples to save computational complexity. However, the question of how one practically chose the values of N_i's remains. The choices of the value of the N_i's are govern by the frequency error variance of the estimate from the individual ith passes. If the frequency estimate, $\hat{f}_m$, of the ith was too far away from actual frequency, it will cause the frequency estimator converge on an incorrect frequency. In fact, in the original algorithm presented in [1], the initial frequency estimate error must be bound between the frequency that is represented by $\pm 1/2$ of a DFT bin to ensure convergence of the algorithm. Since we are increasing the number of samples, N_i, at each pass, one must ensure the frequency estimate error of the ith pass must be smaller than $\pm 1/2$ the frequency represented by a DFT bin of the next pass. This relationship can be mathematically represented as

$$\sqrt{\frac{N_i \sin^2\left(\frac{\pi}{2N_i}\right)\tan^2\left(\frac{\pi}{2N_i}\right)}{4\rho\pi^2}} \leq \frac{1}{2N_{i+1}}. \tag{21}$$

In addition to the above condition, the value of N_0, which corresponds to the length of initial FFT for the initial peak search, has an extra constraint in the form of the performance threshold at low SNR as discussed in [3]. The performance threshold has to do with the nonlinear nature of the frequency estimation problem as low SNR. A detail discussion on the relationship between performance threshold, SNR and the number of samples is out of the scope of this paper. There are a number of papers that addresses this issue and readers are recommended to look at references [9–12] for detail analysis of performance threshold. Only the findings from [9] are discussed in this paper because of the simplistic nature of the results and the ease of applying them to determine the optimum value of N_0 given it has to operate above certain SNR.

In [9], the author derived an approximate threshold indicator given by

$$E\left(N\delta^*\right)^2 = \frac{3}{2\rho N}, \tag{22}$$

where δ^* is the approximation of the normalised frequency error. It was found there is a relationship between the indicator quantity given in Eq. (22) and the mean square (MS) phase error given by

$$E\left(N\delta^*\right)^2 = \frac{3}{2\rho N} = \frac{3}{4} E\left(\tilde{\theta}\right)^2. \tag{23}$$

The author in [9] also found the MS phase error associated with the threshold is roughly $E\left(\tilde{\theta}\right)^2 = 0.0625\ \mathrm{rad}^2$. Table 1 was built using Eq. (23) at common values of N.

Table 1
SNR threshold for various values of N

N	Threshold SNR [dB]
1024	−15.05
512	−12.04
256	−9.03
128	−6.02
64	−3.01
32	0

Since these threshold values are approximations, one should allow a operating margin of at least 1.5 dB when deciding upon the value of N_0. For example, if the requirement is for the estimator to operate at −3 dB, one would choose $N_0 = 128$ over $N_0 = 64$ to guarantee proper operation at −3 dB. Compare the values in Table 1 to the relationship as stated in Eq. (21), one can conclude the performance threshold will dominate in deciding on the value of N_0.

5. Simulation results

Figures 2, 3 and 4 shows the simulated results of the error variance performance as a function of SNR for the fre-

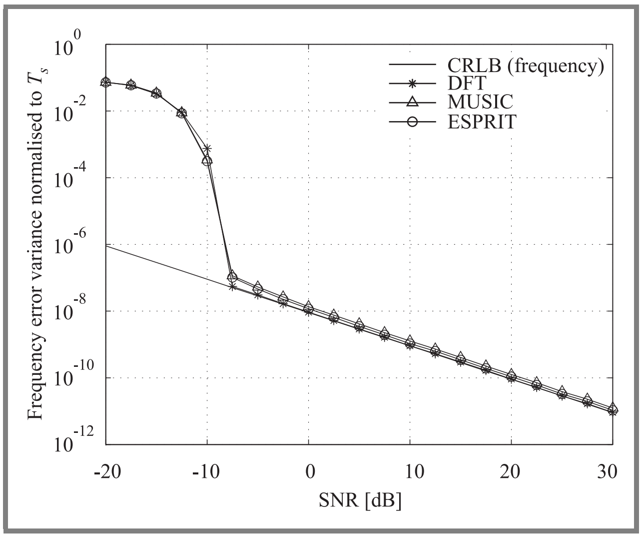

Fig. 2. The error variance of the DFT-based frequency estimator as a function of SNR.

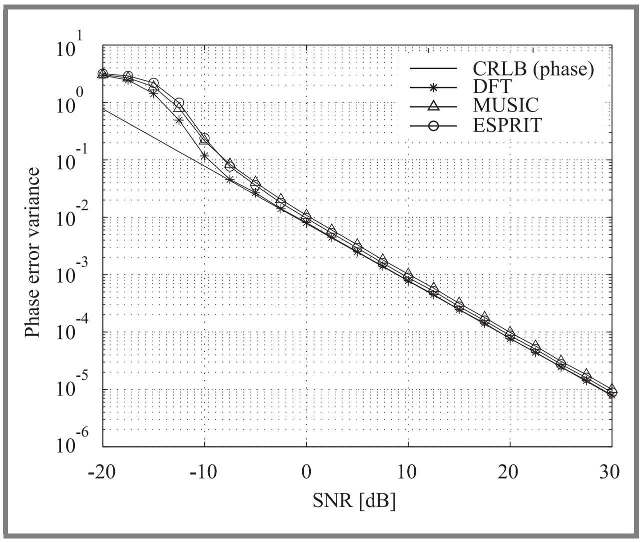

Fig. 3. The error variance of the phase estimator as a function of SNR.

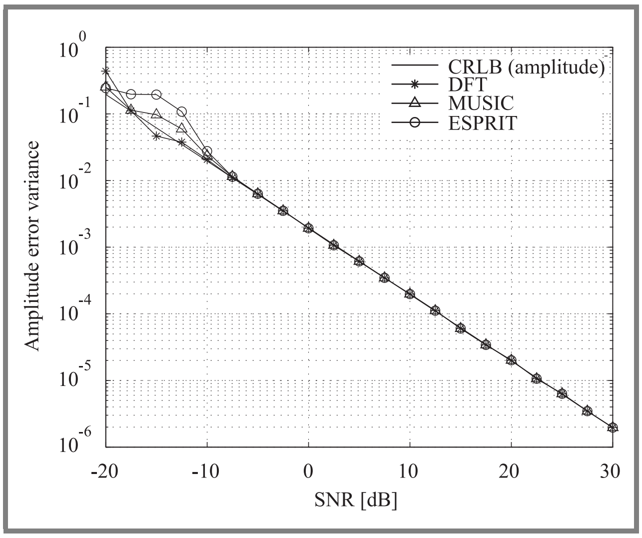

Fig. 4. The error variance of the amplitude estimator as a function of SNR.

quency, phase and amplitude estimator described in Section 2 compared to results obtained by the Matlab^{TM} "rootmusic" algorithm and TLS-ESPRIT algorithm presented in [13]. The number of data points used in the simulation equals to 256 and the size of the autocorrelation matrix used for the "rootmusic" and TLS-ESPRIT algorithm equals to 64. For each trial a random frequency and phase is generated from two independent uniform distributions with the range $-f_s/2$ to $f_s/2$ and 0 to 2π, respectively. The results shown are obtained by averaging over 6000 trials. It can be seen that the simulated results obtained by the proposed algorithm is on par with those obtained by using "rootmusic" and TLS-ESPRIT and yet computationally very efficient.

Figure 5 shows a comparison of the frequency estimation error variance as a function of SNR between the modified

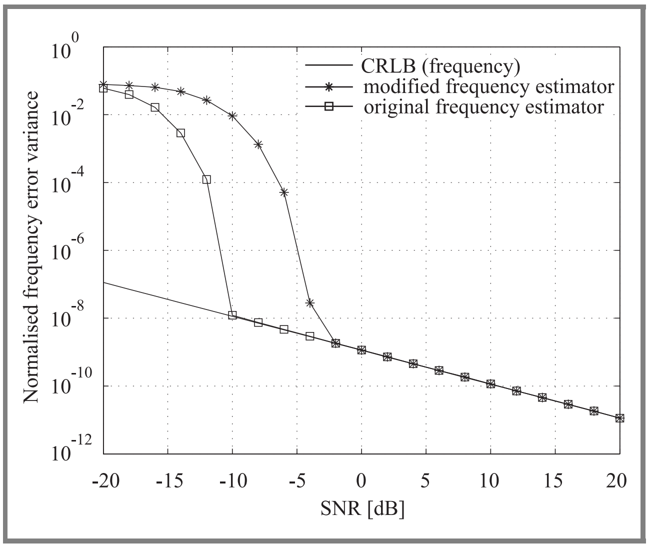

Fig. 5. The error variance of the modified frequency estimator compared to the original frequency estimator.

frequency estimator as described in Section 4 against the frequency estimator as described in Subsection 2.1. The frequency estimate obtained from two passes of the modified frequency estimator with $N_0 = 128$ and $N_1 = 512$. The number of samples was kept constant at $N = 512$ for the frequency estimator as described in Subsection 2.1. The results shown are obtained by averaging over 6000 trials. As expected, the modified version of the frequency estimator's performance threshold is at a higher SNR than the original estimator. However the modified frequency estimator only required $N_0 \log_2 N_0 + 4N_0 + 4N_1 = 3456$ complex multiplications which is approximately half of what is required for the original frequency estimator which equals to $N \log_2 N + 4N = 6656$ complex multiplications.

6. Conclusion

Expanding upon [1], a new algorithm for joint frequency, phase and amplitude estimation of a complex exponential has been presented. It was shown that given sufficient number of samples, at high SNR the variance of the phase estimator approaches the asymptotic limit of 0.0475 dB above the CRLB. It was also shown that the modified version of the frequency estimator can significantly reduce the computation complexity with compromising the overall error variance performance except increasing the operation SNR threshold. With the advantage of being computationally efficient, this type of joint frequency and phase estimator is well suited for real time application such as timing and carrier synchronization.

Appendix

Variance of the DFT-based phase estimator

The arctangent of $r_{A,\theta}[n]$ as defined in Eq. (6) was given as

$$\hat{\theta}_{c,\hat{f}_m} = \tan^{-1}\left(\frac{\sum_{n=0}^{N-1} A\sin(2\pi\varepsilon n T_s + \theta_c) + \eta_s}{\sum_{n=0}^{N-1} A\cos(2\pi\varepsilon n T_s + \theta_c) + \eta_c}\right). \quad (24)$$

From [1], ε the frequency estimation error is a random variable with zero mean and variance of

$$Var[\varepsilon] = \frac{N\sin^2\left(\frac{\pi}{2N}\right)\tan^2\left(\frac{\pi}{2N}\right)}{4\pi^2 SNR}. \quad (25)$$

Again, the variance of $\hat{\theta}_{c,\hat{f}_m}$ in Eq. (24) can be found by using the technique of linearization of function of random variables presented in [8]. From the structure of the discriminate D_m defined in Eq. (4) one can conclude ε and η_c are uncorrelated and ε and η_s are uncorrelated. Expanding Eq. (24) in a three dimensional Taylor series expansion with respect to these variable gives

$$Var\left[\hat{\theta}_{c,\hat{f}_m}\right] = \left(\frac{\partial\overline{\hat{\theta}_{c,\hat{f}_m}}}{\partial\varepsilon}\right)^2 Var[\varepsilon] + \left(\frac{\partial\overline{\hat{\theta}_{c,\hat{f}_m}}}{\partial\eta_c}\right)^2 Var[\eta_c] + \left(\frac{\partial\overline{\hat{\theta}_{c,\hat{f}_m}}}{\partial\eta_s}\right)^2 Var[\eta_s], \quad (26)$$

where

$$\left(\frac{\partial\overline{\hat{\theta}_{c,\hat{f}_m}}}{\partial\varepsilon}\right) = \left(\frac{\partial\hat{\theta}_{c,\hat{f}_m}}{\partial\varepsilon}\right)\Bigg|_{\varepsilon=E[\varepsilon],\,\eta_c=E[\eta_c],\,\eta_s=E[\eta_s]} = \pi(N-1), \quad (27)$$

$$\left(\frac{\partial\overline{\hat{\theta}_{c,\hat{f}_m}}}{\partial\eta_c}\right) = \left(\frac{\partial\hat{\theta}_{c,\hat{f}_m}}{\partial\eta_c}\right)\Bigg|_{\varepsilon=E[\varepsilon],\,\eta_c=E[\eta_c],\,\eta_s=E[\eta_s]} = -\frac{\sin(\theta)}{NA}, \quad (28)$$

$$\left(\frac{\partial\overline{\hat{\theta}_{c,\hat{f}_m}}}{\partial\eta_s}\right) = \left(\frac{\partial\hat{\theta}_{c,\hat{f}_m}}{\partial\eta_s}\right)\Bigg|_{\varepsilon=E[\varepsilon],\,\eta_c=E[\eta_c],\,\eta_s=E[\eta_s]} = \frac{\cos(\theta)}{NA}. \quad (29)$$

Solving Eq. (26) yields

$$Var\left[\hat{\theta}_{c,\hat{f}_m}\right] = \frac{N^2(N-1)^2\sin^2\left(\frac{\pi}{2N}\right)\tan^2\left(\frac{\pi}{2N}\right) + 2}{4N\,SNR}. \quad (30)$$

Acknowledgements

The authors would like to thank the Australian Cooperative Research Center for Satellite Systems and the University of Technology, Sydney, for their funding and support. The work was funded in part by the Commonwealth Government of Australia through the Cooperative Research Centers Program.

References

[1] S. Reisenfeld, "A highly accurate algorithm for the estimation of the frequency of a complex exponential in additive Gaussian noise", in *5th Austr. Commun. Theory Worksh.*, Newcastle, Australia, 2004, pp. 154–158.

[2] S. Reisenfeld and E. Aboutanios, "A new algorithm for the estimation of the frequency of a complex exponential in additive Gaussian noise", *IEEE Commun. Lett.*, vol. 7, issue 11, pp. 529–551, 2003.

[3] D. C. Rife and R. Boorstyn, "Single tone parameter estimation from discrete-time observations", *IEEE Trans. Inform. Theory*, vol. IT-20, no. 5, pp. 591–598, 1974.

[4] S. Kay, "A fast and accurate signal frequency estimator", *IEEE Trans. Acoust., Speech Sig. Proces.*, vol. 37, no. 12, pp. 1987–1990, 1989.

[5] R. Schmidt, "Multiple emitter location and signal parameter estimation", *IEEE Trans. Anten. Propagat.*, vol. 34, no. 3, pp. 276–290, 1986.

[6] R. Roy and T. Kailath, "ESPRIT – estimation of signal parameters via rotational invariance techniques", *IEEE Trans. Acoust., Speech Sig. Proces.*, vol. 37, no. 7, pp. 984–995, 1989.

[7] S. M. Kay, *Fundamentals of Statistical Signal Processing Estimation Theory*. Upper Saddle River: Prentice Hall, 1993.

[8] *Problems in Probability Theory, Mathematical Statistics and Theory of Random Functions*, A. A. Sveshnikov, Ed. New York: Dover, 1968.

[9] B. James, B. D. O. Anderson, and R. C. Williamson, "Characterization of threshold for single tone maximum likelihood frequency estimation", *IEEE Trans. Sig. Proces.*, vol. 43, pp. 817–821, 1995.

[10] A. O. Steinhardt and C. Bretherton, "Thresholds in frequency estimation", in *Proc. ICASSP*, Tampa, USA, 1985, vol. 10, pp. 1273–1276.

[11] B. G. Quinn and P. J. Kootsookos, "Threshold behavior of the maximum likelihood estimator of frequency", *IEEE Trans. Sig. Proces.*, vol. 42, pp. 3291–3294, 1994.

[12] L. Knockaert, "The Barankin bound and threshold behavior in frequency estimation", *IEEE Trans. Sig. Proces.*, vol. 45, pp. 2398–2401, 1997.

[13] D. G. Manolakis, V. K. Ingle, and S. M. Kogon, *Statistical and Adaptive Signal Processing – Spectral Estimation, Signal Modelling, Adaptive Filtering and Array Processing*. Boston: McGraw Hill, 2000.

Jeffrey Tsui received his B.E. degree in telecommunication engineering the University of Technology, Sydney, in 2002. He is currently a final stage Ph.D. student at the University of Technology, Sydney. His research interests include signal processing and its applications in the areas of satellite communication and wireless communication technologies.

e-mail: jtsui@eng.uts.edu.au
Faculty of Engineering
Cooperative Research Center for Satellite Systems
University of Technology, Sydney
PO Box 123, Broadway, NSW 2007, Australia

Sam Reisenfeld received the B.Sc. degree in information engineering specializing in communication systems engineering from the University of Illinois in 1969 and the M.Sc. and Ph.D. degrees in communication systems engineering from UCLA in 1972 and 1979, respectively. From June, 1969 until September, 1988, he was a space communication systems engineer at the Hughes Aircraft Company, Systems Laboratories, Space and Communications Group, El Segundo, California. From September, 1988 until the present, he is an Associate Professor of telecommunication engineering at the University of Technology, Sydney, Australia. From January, 1997 until the present, he also is the Program Director for Ka band Satellite Communications Research and Development in the Australian Cooperative Research Centre for Satellite Systems. He is a Member of Phi Kappa Phi and the IEEE. He received the IEEE Millennium Medal in November, 2000.

e-mail: samr@uts.edu.au
Faculty of Engineering
Cooperative Research Center for Satellite Systems
University of Technology, Sydney
PO Box 123, Broadway, NSW 2007, Australia